New Discrete States in Two-Dimensional Supergravity

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Abstract

Two-dimensional string theory is known to contain the set of discrete states that are the SU(2) multiplets generated by the lowering operator of the $SU(2)$ current algebra. Their structure constants are defined by the area preserving diffeomorphisms in two dimensions. In this paper we show that the interaction of $d = 2$ superstrings with the super-conformal $\beta - \gamma$ ghosts enlarges the actual algebra of the dimension 1 currents and hence the new ghost-dependent discrete states appear. Generally, these states are the SU(N) multiplets, if the algebra includes the currents of ghost numbers n : $-N \leq n \leq N - 2$, not related by picture-changing. We compute the structure constants of these ghost-dependent discrete states for $N = 3$ and express them in terms of $SU(3)$ Clebsch-Gordan coefficients, relating this operator algebra to the volume preserving diffeomorphisms in $d = 3$. For general N, the operator algebra is conjectured to be isomorphic to $SDiff(N)$. This points at possible holographic relations between two-dimensional superstrings and field theories in higher dimensions. **PACS:**$04.50. + h; 11.25.Mj$.

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Introduction

The spectrum of physical states of non-critical one-dimensional string theory (or , equivalently, of critical string theory in two dimensions) is known to contain the "discrete states" of non-standard b-c ghost numbers 0 and 2 (while the standard unintegrated vertex operators always carry the ghost number 1) [1], [2],[3],[4],[5] These states appear at the special (integer or half integer) values of the momentum p but are absent for generic p [1]

The appearance of these states is closely related to the $SU(2)$ symmetry, emerging when the theory is compactified on a circle with the self-dual radius [2], [3]

The $SU(2)$ symmetry is generated by 3 currents of conformal dimension 1:

$$T_3 = \oint \frac{dz}{2i\pi} \partial X(z)$$
$$T_+ = \oint \frac{dz}{2i\pi} e^{iX\sqrt{2}} \tag{1}$$
$$T_- = \oint \frac{dz}{2i\pi} e^{-iX\sqrt{2}}$$

The tachyonic operators $W_s = e^{\frac{isX}{\sqrt{2}}}$ with non-negative integer s are the highest weight vestors in the representation of the "angular momentum" s (here X is the $c = 1$ matter field).

The discrete primaries are then constructed by repeatedly acting on W_s with the lowering operator T_- of SU(2). For example, acting on W_s n times with T_- one generally produces the primary fields of the form $W_{s,n} = P(\partial X, \partial^2 X, ...)e^{i(\frac{s}{\sqrt{2}} - n\sqrt{2})X}$ with $n < s$, where P is some polynomial in the derivatives of X, straightforward but generally hard to compute. After the Liouville dressing and the multiplication by the ghost field c, $W_{s,n}$ becomes the dimension 0 primary field $cW_{s,n}$, i.e. the BRST-invariant operator for a physical state. The set of operators $cW_{s,n}$ thus consists of the multiplet states of $SU(2)$ which OPE structure constants can be shown to form the enveloping of $SU(2)$ [2]. The discrete operators of non-standard adjacent ghost numbers 0 and 2 $Y_{s,n}^0$ and $Y_{s,n}^{+2}$ can be obtained from $cW_{s,n}$ simply by considering the BRST commutators [3]: $\{Q_{BRST}, Y_{s,n}^0\} = cW_{s,n}$ and $\{Q_{BRST}, cW_{s,n}\} = Y_{s,n}^{+2}$. One has to evaluate these commutators for arbitrary s first and and then take s to be the appropriate integer number. For arbitrary s the ghost number 0 and 2 Y-operators are respectively BRST non-invariant and exact, but

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[1] in our paper the stress tensor for the X-field is normalized as $T = -\frac{1}{2}(\partial X)^2$ which is one half of the normalization used in [2] hence only the integer values of p are allowed

for the integer values of s they become independent physical states, defining the BRST cohomologies with non-trivial $b - c$ ghost numbers. At the first glance, things naively could seem to be the same in case when the $c = 1$ theory is supersymmetrized on the worldsheet. However, in this paper we will show that the interaction of the $c = 1$ theory with the system of $\beta - \gamma$ ghosts dramatically extends the spectrum of the physical discrete states. Consider the $c = 1$ model supersymmetrized on the worldsheet coupled to the super Liouville field. The worldsheet action of the system in the conformal gauge is given by [6]

$$S = S_{X-\psi} + S_L + S_{b-c} + S_{\beta-\gamma}$$

$$S_{X-\psi} = \frac{1}{4\pi} \int d^2 z \{\partial X \bar{\partial} X + \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}\}$$

$$S_L = \frac{1}{4\pi} \int d^2 z \{\partial \varphi \bar{\partial} \varphi + \lambda \bar{\partial} \lambda + \bar{\lambda} \partial \bar{\lambda} - F^2 + 2\mu_0 b e^{b\varphi}(ib\lambda\bar{\lambda} - F)\} \tag{2}$$

$$S_{b-c} + S_{\beta-\gamma} = \frac{1}{4\pi} \int d^2 z \{b\bar{\partial} c + \bar{b}\partial \bar{c} + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma}\}$$

with $Q \equiv b + b^{-1}$ being the background charge.

The stress tensors of the matter and the ghost systems and the standard bosonization relations for the ghosts are given by

$$T_m = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}\partial \psi \psi - \frac{1}{2}(\partial \varphi)^2 + \frac{Q}{2}\partial^2 \varphi$$

$$T_{gh} = \frac{1}{2}(\partial \sigma)^2 + \frac{3}{2}\partial^2 \sigma + \frac{1}{2}(\partial \chi)^2 + \frac{1}{2}\partial^2 \chi - \frac{1}{2}(\partial \phi)^2 - \partial^2 \phi \tag{3}$$

$$c = e^{\sigma}, b = e^{-\sigma}$$

$$\gamma = e^{\phi-\chi}, \beta = e^{\chi-\phi}\partial \chi$$

The central charge of the super Liouville system is equal to $c_L = 1 + 2Q^2$ and, since $c_{X-\psi} + c_L + c_{b-c} + c_{\beta-\gamma} = 0$, we have $Q^2 = 4$ for one-dimensional non-critical superstrings.

The $SU(2)$ algebra is now generated by the dimension 1 currents

$$T_{0,0} = \oint \frac{dz}{2i\pi} \partial X$$

$$T_{0,1} = \oint \frac{dz}{2i\pi} e^{iX} \psi \tag{4}$$

$$T_{0,-1} = \oint \frac{dz}{2i\pi} e^{-iX} \psi$$

Here and elsewhere the first lower index refers to the superconformal ghost number and the second to the momentum in the X direction. In comparison with the bosonic case,

a crucial novelty emerges due to the interaction of the matter with the $\beta - \gamma$ system of superconformal ghosts. That is, we will show that, apart from the dimension one currents (4) generating SU(2),there is also a set of BRST-nontrivial dimension 1 currents mixed with $\beta - \gamma$ ghosts, i.e. those existing at nonzero $\beta - \gamma$ ghost pictures and not reducible to (4) by any picture-changing transformation. These extra currents turn out to enhance the actual underlying symmetry of the theory from $SU(2)$ two $SU(n)$ with $n \geq 3$ being the highest order ghost number cohomology of the operators in the current algebra (see the discussion below). For example, the currents of the form

$$T_{-n,n-1} = \oint \frac{dz}{2i\pi} e^{-n\phi+(n-1)X} \psi(z) \tag{5}$$

with ghost numbers $n \leq -3$ are BRST-nontrivial, even though they are annihilated by picture-changing transformation and no picture zero version of these currents exists. The possibility of existence of BRST-nontrivial operators annihilated by picture-changing has been discussed in [7] in case of $n = -3$ for the critical $D = 10$ superstrings. In this paper we shall also present a proof of the BRST non-triviality of the dimension 1 currents with arbitrary negative ghost numbers $n \leq -3$ in two-dimensional critical superstring theory. Next, acting on $T_{-n,n-1}$ repeatedly with T_- of (4) one obtains the currents of the form

$$T_{-n,m} = \oint \frac{dz}{2i\pi} P_{-n,m}(\partial X, \partial^2 X, ...; \psi, \partial \psi, ...) e^{-n\phi+imX}$$
$$|m| \leq n - 1 \tag{6}$$

which generally are the BRST non-trivial Virasoro primaries of ghost number $-n$ annihilated by picture-changing. Here $P_{-n,m}$ are the polynomials in $\partial X, \psi$ and their derivatives, with the conformal weight $h = \frac{1}{2}(n^2 - m^2) + n + 1$ so that the overall dimension of the integrands in (6) is equal to 1. In this paper we will derive the precise expressions for these polynomials for the cases $n = -3$ and $n = -4$. So if one considers all the picture-inequivalent currents with ghost numbers $p = 0, -3, ... - n$, including those of (4), (all the dimension 1 currents with ghost numbers -1 and -2 are equivalent to the $SU(2)$ generators (4) by picture-changing) one has the total number $n^2 - 1$ of T-generators (6). In this paper we will show (precisely for $|n| \leq 4$ and conjecture for larger values of $|n|$) that the algebra of operators $T_{p,m}$ $|m| \leq |p - 1|, p = -3, ..., -n$ combined with 3 $SU(2)$ generators of (4) is simply $SU(n)$ with the generators $T_{0,m}$ $(m = 0; -3, -4, ... - n)$ being in the Cartan

subalgebra of $SU(n)$. The next step is easy to guess. For each given n one starts with the Liouville dressed tachyonic Virasoro primaries

$$\oint \frac{dz}{2i\pi} V_l = \oint \frac{dz}{2i\pi} e^{ilX+(l-1)\varphi}(l\psi - i(l-1)\varphi)$$

, with integer l and acts on them with various combinations of lowering T-operators (i.e. those having singular OPEs with integrands of V_l). The obtained operators will be the multiplets of $SU(n)$, including the operators of BRST cohomologies with non-trivial ghost dependence (not removable by the picture changing). These operators are generally of the form

$$V_{q,n;m,l} = \oint \frac{dz}{2i\pi} : e^{(l-1)\varphi-q\phi+imX} P_{q,n;m,l}(\partial X, \partial^2 X, ...\psi, \partial\psi, ..., \partial\phi, \partial^2\phi...) : (z) \tag{7}$$

$$|m| \leq l$$

Of course, the usual well-known discrete states form the operator subalgebra of ghost cohomology number zero in the space of $\{V_{q,n;m,l}\}$. Here and elsewhere the term "ghost cohomology" refers to the factorization over all the picture-equivalent states; thus the ghost cohomology of number $-q$ ($q > 0$) consists of BRST invariant and non-trivial operators with the ghost numbers $-r \leq -q$ existing at the maximal picture $-q$, so that their picture $-q$ expressions are annihilated by the operation of the picture-changing; conversely, the operators of a positive ghost cohomology number $q > 0$ are the physical operators annihilated by the inverse picture changing at the picture q, so that they don't exist at pictures below q. Of course, the ghost cohomologies are not cohomologies in the literal sense, since the picture-changing operator is not nilpotent. In the rest of the paper we will particularly demonstrate the above construction by precise computations, deriving expressions for the new vertex operators of the ghost-dependent discrete states and computing their structure constants.

Extended current algebra and ghost-dependent discrete states

As usual, in the supersymmetric case three SU(2) currents (4) can be taken at different ghost pictures. For example, the picture -1 expressions for the currents (4) : $e^{-\phi}\psi, e^{-\phi\pm iX}$ are the only dimension 1 generators at this superconformal ghost number; similarly, all the ghost number -2 dimension 1 operators: $e^{-2\phi}\partial X, e^{-2\phi\pm iX}\psi$ are just the the $SU(2)$ generators (4) at picture -2. However, at ghost pictures of -3 and below, the new dimension

1 generators, not reducible to (4) by picture-changing, appear. The first example is the generator given by the worldsheet integral

$$T_{-3,2} = \oint \frac{dz}{2i\pi} e^{-3\phi+2iX} \psi(z) \tag{8}$$

where as usual ϕ is a bosonized superconformal ghost with the stress tensor $T_\phi = -\frac{1}{2}(\partial\phi)^2 - \partial^2\phi$. This operator is annihilated by the picture-changing transformation, and we leave the proof of its BRST non-triviality until the next section. Given the generator (8), it is now straightforward to construct the dimension 1 currents of the ghost number -3 with the momenta $0, \pm 1$ and -2, which are the Virasoro primaries, not related to (4) by picture-changing. For instance this can be done by taking the lowering operator $T_{0,-1} = \oint \frac{dz}{2i\pi} e^{-iX} \psi(z)$ of $SU(2)$ and acting on (8). Performing this simple calculation, we obtain the following extra five generators in the ghost number -3 cohomology:

$$T_{-3,2} = \oint \frac{dz}{2i\pi} e^{-3\phi+2iX} \psi(z)$$

$$T_{-3,1} = \oint \frac{dz}{2i\pi} e^{-3\phi+iX} (\partial\psi\psi + \frac{1}{2}(\partial X)^2 + \frac{i}{2}\partial^2 X)(z)$$

$$T_{-3,-1} = \oint \frac{dz}{2i\pi} e^{-3\phi-iX} (\partial\psi\psi + \frac{1}{2}(\partial X)^2 - \frac{i}{2}\partial^2 X)(z) \tag{9}$$

$$T_{-3,0} = \oint \frac{dz}{2i\pi} e^{-3\phi} (\partial^2 X\psi - 2\partial X\partial\psi)(z)$$

$$T_{-3,-2} = \oint \frac{dz}{2i\pi} e^{-3\phi-2iX} \psi(z)$$

It is straightforward to check that all these generators are the primary fields commuting with the BRST charge. The next step is to show that the operators (9) taken with 3 standard SU(2) generators $T_{0,0}, T_{0,1}$ and $T_{0,-1}$ of (4) combine into 8 generators of $SU(3)$ (up to picture-changing transformations), with $T_{0,0}$ and $T_{-3,0}$ generating the Cartan subalgebra of $SU(3)$. Here an important remark should be made. Straightforward computation of the commutators of some of the generators (9) can be quite cumbersome due to high order singularities in the OPE's involving the exponential operators with negative ghost numbers as $e^{\alpha\phi}(z)e^{\beta\phi}(w) \sim (z-w)^{-\alpha\beta}e^{-(\alpha+\beta)\phi}(w) + ...$ For instance, the computation of the commutator of $T_{-3,2}$ with $T_{-3,-2}$ would involve the OPE $e^{-3\phi+2iX}(z)e^{-3\phi-2iX}(w) \sim (z-w)^{-13}e^{-6\phi} + ...$, so the OPE coefficient in fromt of the single pole would be cumbersome to compute. In addition, as the ghost picture of the r.h.s. of the commutator is equal to -6 one would need an additional picture-changing transformation to relate it to the T-operators (4),(9). Things, however, can be simplified if we

note that the operators $T_{-n,m}$ of ghost cohomology $-n$ ($n = 3, 4, ...$) are picture-equivalent to the appropriate operators from the *positive* ghost number $n - 2$ cohomologies. That is, if in the expressions for any of these operators one replaces $e^{-n\phi}$ with $e^{(n-2)\phi}$ (which has the same conformal dimension as $e^{-n\phi}$) and keeps the matter part unchanged, one gets an equivalent BRST-invariant and nontrivial operator, in the sense that replacing one operator with another inside any correlator does not change the amplitude. Such an equivalence is generally up to certain b-c ghost terms needed to preserve the BRST-invariance of the operators with positive superconformal ghost numbers; however, these terms, generally do not contribute to correlators due to the $b - c$ ghost number conservation conditions [8]. Thus, for example, one can replace

$$T_{-3,2} = \oint \frac{dz}{2i\pi} e^{-3\phi + 2iX} \psi \rightarrow T_{+1,2} = \oint \frac{dz}{2i\pi} e^{\phi + 2iX} \psi \tag{10}$$

(up to the b-c ghost terms). The picture-equivalence of these operators means that, even though they are not straightforwardly related by picture-changing transformations (as $T_{-3,2}$ and $T_{+1,2}$ are respectively annihilated by direct and inverse picture-changings), any correlation functions involving these vertex operators are equivalent under the replacement (10). The precise relation between these operators is given by

$$T_{+1,2} = Z(:: \Gamma^4 : ce^{-3\phi + 2iX} :) \tag{11}$$

where $: \Gamma :^4$ is the normally ordered fourth power of the usual picture-changing operator Γ while the Z-transformation mapping the local vertices to integrated is the analogue of Γ for the $b - c$ ghosts [8]. The Z-transformation can be performed by using the BRST-invariant non-local Z-operator of $b - c$ ghost number -1; the precise expression for this operator was derived in [8] and is given by

$$Z = \oint_w \frac{du}{2i\pi} (u - w)^3 (bT + 4ce^{2\chi - 2\phi} T^2)(u)$$

where T is the full matter + ghost stress-energy tensor; the integral is taken over the worldsheet boundary and acts on local operators at w. Similarly, for any ghost number $-n$ generators one can replace

$$T_{-n,m} \rightarrow T_{n-2,m}; n \leq -3 \tag{12}$$

Since the Liouville and the $\beta - \gamma$ stress tensors both have the same Q^2, the equivalence relations (10), (12) can formally be thought of as the special case of the reflection identities

[9],[10] in $d = 2$ super Liouville theory in the limit of zero cosmological constant μ_0. Using the equivalence identities (10),(12) and evaluating the OPE simple poles in the commutators it is now straightforward to determine the algebra of operators (4),(9). At this point, it is convenient to redefine:

$$L = \frac{i}{2}T_{0,0}, H = \frac{i}{3\sqrt{2}}T_{-3,0}$$

$$G_+ = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,1} + T_{-3,1}), G_- = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,1} - T_{-3,1})$$

$$F_+ = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,-1} + T_{-3,-1}), F_- = \frac{1}{2\sqrt{2}}(\sqrt{2}T_{0,-1} - T_{-3,-1}) \tag{13}$$

$$G_3 = \frac{1}{\sqrt{2}}T_{-3,2}, F_3 = \frac{1}{\sqrt{2}}T_{-3,-2}$$

Then the commutators of the operators L and H with each other and with the rest of (13) are given by

$$[L, H] = 0$$

$$[L, G_+] = \frac{1}{2}G_+; [L, G_-] = \frac{1}{2}G_-; [L, F_+] = -\frac{1}{2}F_+; [L, F_-] = -\frac{1}{2}F_-$$

$$[L, G_3] = G_3; [L, F_3] = -F_3 \tag{14}$$

$$[H, G_+] = -G_+; [H, G_-] = G_-; [H, F_+] = F_+; [H, F_-] = -F_-$$

$$[H, G_3] = [H, F_3] = 0$$

i.e. L and H indeed are in the Cartan subalgebra; the remaining commutators of the currents(13) are then easily computed to give

$$[G_3, G_+] = 0; [G_3, G_-] = 0; [G_3, F_+] = G_-; [G_3, F_-] = -G_+$$

$$[F_3, F_+] = 0; [F_3, F_-] = 0; [F_3, G_+] = F_-; [F_3, G_-] = -F_+$$

$$[F_-, F_+] = -F_3; [G_-, G_+] = -G_3; [G_-, F_+] = [F_-, G_+] = 0 \tag{15}$$

$$[F_+, G_+] = L - \frac{3}{2}H; [F_-, G_-] = L + \frac{3}{2}H; [G_3, F_3] = 2L$$

Thus the operators $L, H, F_\pm, G_\pm, F_3$ and G_3 of (13) simply define the Cartan-Weyl basis of $SU(3)$. Their BRST invariance ensures that this $SU(3)$ algebra intertwines with the superconformal symmetry of the theory. Therefore, just like in the case of usual $SU(2)$ discrete states of two-dimensional supergravity, we can generate the extended set of discrete super Virasoro primaries by taking a dressed tachyonic exponential operator

$$V = \oint \frac{dz}{2i\pi} e^{ilX+(l-1)\varphi}(l\psi - i(l-1)\lambda)$$

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at integer values of l and acting with the generators of the lowering subalgebra of $SU(3)$ ($F_\pm$ and F_3). The obtained physical operators are then the multiplets of SU(3) and include the new discrete physical states of non-trivial ghost cohomologies, not reducible to the usual $SU(2)$ primaries by any picture-changing transformations. Such a construction will be demonstrated explicitly in the next section. The scheme explained above can be easily generalized to include the currents of higher values of ghost numbers. For example, to include the currents of ghost numbers up to -4 (or up to $+2$, given the equivalence relations(12)) one has to start with the generator $T_{-4,3} = \oint \frac{dz}{2i\pi} e^{-4\phi+3iX}\psi$. The conformal dimension of the integrand is equal to 1. This generator is not related to any of the operators (4),(9) by the picture-changing and therefore is the part of the ghost number -4 cohomology. This cohomology contains 7 new currents $T_{-4,m}, |m| \leq 3$, which are the BRST-invariant super Virasoro primaries. As previously, these currents can be generated from $T_{-4,3}$ by acting on it repeatedly with $T_{0,-1}$ of (4).The resulting operators are given by

$$T_{-4,\pm3} = \oint \frac{dz}{2i\pi} e^{-4\phi\pm3iX}\psi(z)$$

$$T_{-4,2} = \oint \frac{dz}{2i\pi} e^{-4\phi+2iX}(\frac{1}{2}\partial^2\psi\psi - \frac{i}{6}\partial^3 X + \frac{i}{6}(\partial X)^3 - \frac{1}{2}\partial X\partial^2 X)$$

$$T_{-4,1} = \oint \frac{dz}{2i\pi} e^{-4\phi+iX}(\frac{1}{2}\psi\partial\psi\partial^2\psi + \frac{1}{24}P^{(4)}_{-iX}\psi - \frac{1}{4}P^{(2)}_{-iX}\partial^2\psi - \frac{1}{4}(P^{(2)}_{-iX})^2\psi)$$

$$T_{-4,-1} = \oint \frac{dz}{2i\pi} e^{-4\phi-iX}(\frac{1}{2}\psi\partial\psi\partial^2\psi + \frac{1}{24}P^{(4)}_{iX}\psi - \frac{1}{4}P^{(2)}_{iX}\partial^2\psi - \frac{1}{4}(P^{(2)}_{iX})^2\psi)$$

$$T_{-4,-2} = \oint \frac{dz}{2i\pi} e^{-4\phi-2iX}(\frac{1}{2}\partial^2\psi\psi + \frac{i}{6}\partial^3 X - \frac{i}{6}(\partial X)^3 - \frac{1}{2}\partial X\partial^2 X)$$

$$T_{-4,0} = \oint \frac{dz}{2i\pi} e^{-4\phi}\{2i\partial X\partial\psi\partial^2\psi + P^{(2)}_{-iX}\psi\partial^2\psi - \frac{2}{3}P^{(3)}_{-iX}\psi\partial\psi$$

$$-\frac{1}{6}P^{(3)}_{-iX}P^{(2)}_{-iX} + (\partial X)^2\psi\partial^2\psi + \frac{7i}{8}\partial X P^{(4)}_{-iX}$$

$$-i(\partial X)^3\psi\partial\psi - \frac{i}{2}\partial X P^{(2)}_{-iX}\psi\partial\psi + \frac{i}{4}\partial X(P^{(2)}_{-iX})^2 - \frac{1}{4}(\partial X)^2 P^{(3)}_{-iX}\}$$

(16)

Here $P^{(n)}_{\pm iX}; n = 2, 3, 4$ are the conformal weight n polynomials in the derivatives of X defined as

$$P^{(n)}_{f(X(z))} = e^{-f(X(z))}\frac{\partial^n}{\partial z^n}e^{f(X(z))}$$

(17)

for any given function $f(X)$. For example, taking $f = iX$ one has $P^{(1)}_{iX} = i\partial X, P^{(2)}_{iX} = i\partial^2 X - (\partial X)^2$ etc. The special case $f(X(z)) = X(z) = \frac{z^2}{2}$ gives the usual Hermite polynomials in z. The definition (17) can be straightforwardly extended to the functions of n

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variables $f \equiv f(X_1(z), ... X_n(z))$. The direct although lengthy computation of the commutators (using the picture-equivalence relations (12)) shows that seven $T_{-4,n}$-generators of the ghost number -4 cohomology (16) taken with 8 generators of $SU(3)$ of (13) combine into 15 generators of $SU(4)$. As in the case of $SU(3)$ the Cartan subalgebra of $SU(4)$ is generated by the zero momentum currents L, H and $T_{-4,0}$ of (13) and (16). As previously, this $SU(4)$ algebra intertwines with the conformal symmetry of the theory. The repeated applications of the lowering subalgebra of $SU(4)$ to the dressed tachyonic vertex lead to the extended set of the ghost-dependent discrete states which are the multiplets of $SU(4)$. It is natural to assume that this construction can be further generalized to include the generators of higher ghost numbers. The total number of the BRST-invariant generators $T_{-n,m}$ with the ghost numbers $-N \leq -n \leq -3$ and the momenta $-n+1 \leq m \leq n-1$ for each n, combined with three standard $SU(2)$ generators, is equal to N^2-1. It is natural to conjecture that altogether they generate $SU(N)$, although in this paper we leave this fact without a proof. As before, the Cartan subalgebra of SU(N) is generated by the commuting zero momentum generators $T_{-n,0}; n = 0; 3, 4, ..., N$. Applying repeatedly the lowering $SU(N)$ subalgebra (i.e. the $T_{-n,m}$'s with $m \leq 0$) to the dressed tachyon operator would then generate the extended set of the physical ghost-dependent discrete states - the multiplets of SU(N). In the next section we will address the question of BRST-nontriviality of the $T_{-n,m}$-generators. Finally, we will demonstrate the explicit construction of new ghost-dependent discrete states from by $T_{-n,m}$ for the case of $SU(3)$ and compute their structure constants.

BRST nontriviality of the $T_{-n,m}$-currents

The BRST charge of the one-dimensional NSR superstring theory is given by the usual worldsheet integral

$$Q_{brst} = \oint \frac{dz}{2i\pi} \{cT - bc\partial c + \gamma G_{matter} - \frac{1}{4}b\gamma^2\} \qquad (18)$$

where G_{matter} is the full matter ($c = 1$+Liouville) supercurrent. The BRST-invariance of the $T_{-n,m}$-currents is easy to verify by simple calculation of their commutators with Q_{brst}. Indeed, as the primaries of dimension one and $b-c$ ghost number zero they commute with the stress-tensor part of Q_{brst} and their operator products with the supercurrent terms of Q_{brst} are all non-singular. The BRST-invariance of the picture-equivalent currents $T_{n-2,m} = Z(: \Gamma^{2n-2}cS_{-n,m} :)$, where $S_{-n,m}$ are the integrands of $T_{-n,m}$, simply follows from the invariance of Γ and Z.

The BRST-nontriviality of these operators is less transparent and needs separate proof. In principle, the BRST non-triviality of any of the operators can be proven if one shows that they produce non-vanishing correlators, which is what will be done precisely in the next section for the case of $n = 3$. In this section, however, we will present the proof for general values of n, without computing the correlators. For each n, it is sufficient to prove the non-triviality of $T_{-n,n-1}$-operators, as the currents with the lower momenta are obtained from $T_{-n,n-1}$ by repeated applications of the lowering generator of $SU(2)$. The BRST non-triviality of new discrete states the multiplets of $SU(n)$ then automatically follows from the non-triviality of the T-currents. In other words, we need to show that for each n there are no operators W_n in the small Hilbert space such that $[Q_{brst}, W_n] = T_{-n,n-1}$. As it is clear from the form of the BRST charge (18) there are only two possible expressions for W_n (up to total derivatives that drop out after the worldsheet integration) which commutator with the BRST charge may produce the T-currents:

$$W_n = W_n^{(1)} + W_n^{(2)}$$

$$W_n^{(1)} = \sum_{k=1}^{n-1} \alpha_k \oint \frac{dz}{2i\pi} e^{-(n+1)\phi+i(n-1)X} \partial^{(k)} \xi \partial^{(n-k)} X \tag{19}$$

$$W_n^{(2)} = \sum_{\substack{k,l=1,k\neq l}}^{k,l=n;k+l\leq 2n} \alpha_{kl} \oint \frac{dz}{2i\pi} e^{-(n+2)\phi+i(n-1)X} \psi \partial^{(k)} \xi \partial^{(l)} \xi \partial^{(2n-k-l)} c$$

with α_k and α_{kl} being some coefficients and $\xi = e^\chi$. Generically, the W-operators may also contain the worldsheet derivatives of the exponents of ϕ (corresponding to the derivatives of delta-functions of the ghost fields), but one can always bring these operators to the form (19) by partial integration. Clearly, the operators $W_n^{(1)}$ and $W_n^{(2)}$ are the conformal dimension one operators satisfying the relations

$$[\oint \frac{dz}{2i\pi}\gamma^2\psi\partial X, W^{(1)}] \sim T_{-n,n+1}$$

$$[\oint \frac{dz}{2i\pi}\gamma^2 b, W_n^{(2)}] \sim T_{-n,n-1} \tag{20}$$

$$[\oint \frac{dz}{2i\pi}\gamma^2\psi\partial X, W^{(2)}] = [\oint \frac{dz}{2i\pi}\gamma^2 b, W_n^{(1)}] = 0$$

Therefore the T-currents are BRST-trivial if and only if there exists at least one combination of the coefficients α_k or α_{kl} such that W_n commutes with the stress tensor part of Q_{brst}, that is, either

$$[\oint \frac{dz}{2i\pi}(cT - bc\partial c), W_n^{(1)}] = 0 \tag{21}$$

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or

$$[\oint \frac{dz}{2i\pi}(cT - bc\partial c), W_n^{(2)}] = 0 \qquad (22)$$

So our goal is to show that no such combinations exist. We start with $W_n^{(1)}$. Simple computation of the commutator, combined with the partial integration (to get rid of terms with the derivatives of ϕ) gives

$$[\oint \frac{dz}{2i\pi}, W_n^{(1)}] = e^{-(n+1)\phi+i(n-1)X} \sum_{k=1}^{n-1}\sum_{a=1}^{k} \frac{k!}{a!(k-a)!}\partial^{(a)}c\{\partial^{(k-a+1)}\xi\partial^{(n-k)}X\alpha_k$$
$$+\partial^{(n-k)}\xi\partial^{(k-a+1)}X\alpha_{n-k}\} - \alpha_k\{n\partial c\partial^{(k)}\xi\partial^{(n-k)}X + c\partial(\partial^{(k)}\xi\partial^{(n-k)}X) \qquad (23)$$

The operator $W_n^{(1)}$ is BRST-trivial if for any combination of the α_k coefficients this commutator vanishes. The condition $[Q_{brst}, W_n^{(1)}] = 0$ gives the system of linear constraints on α_k. That is,it's easy to see that the right hand side of (23) consists of the terms of the form

$$\sim e^{-(n+1)\phi+i(n-1)X}\partial^{(a)}c\partial^{(b)}\xi\partial^{(n-a-b+1)}X$$

with $a, b \geq 1, a+b \leq n$. The $T_{-n,n-1}$-operator is BRST-trivial if and only if the coefficients (each of them given by some linear combination of α_k) in front of the terms in the right hand side of (23) vanish for each independent combination of a and b. The number of independent combinations of a and b is given by $\frac{1}{2}n(n+1)$ and is equal to the number of constraints on α_k. The number of α_k's is obviously equal to $n-1$ that is, for $n \leq -3$ the number of the constraints is bigger than the number of of α's.This means that the system of equations on α_k has no solutions but the trivial one $\alpha_k = 0; k = 1, ...n - 1$ and no operators of the $W_n^{(1)}$-type satisfying (21) exist. Therefore there is no threat of the BRST-triviality of the $T_{-n,n+1}$-operator from this side. Next, let's consider the case of $W_n^{(2)}$ and show that there is no combination of the coefficients α_{kl} for which (22) is satisfied. The proof is similar to the case of $W_n^{(1)}$. The number of independent coefficients α_{kl} (such that $k, l \geq 1; k < l$ and $k + l \leq 2n$) is equal to $N_1 = n^2 - n$. The commutator $[\oint \frac{dz}{2i\pi}(cT - bc\partial c), W_n^{(2)}]$ leads to the terms of the form

$$\sim \oint \frac{dz}{2i\pi}e^{-(n+2)\phi+i(n-1)X}\psi\partial^{(k)}\xi\partial^{(l)}\xi\partial^{(m)}c\partial^{(2n+1-k-l-m)}c$$

with all integer k, l, m satisfying $k, l \geq 1, m \geq 0; k < l, m < 2n + 1 - k - l - m; k + l + m \leq 2n + 1$. The total number of these independent terms, multiplied by various linear combinations of α_{kl}, is given by the sum $N_2 = 2\sum_{p=1}^{n-1}p(n-p)$ and is equal to the number

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of the linear constraints on α_{kl}. Since for $n \leq -3$ one always has $N_2 > N_1$, there are no non-zero solutions for α_{kl}, and therefore no operators of the $W_n^{(2)}$-type exist. This concludes the proof of the BRST-nontriviality of the T-currents. Our proof implied that, generically, the constraints on α's are linearly independent In fact, such an independence can be demonstrated straightforwardly by the numerical analysis of the systems of linear equations implied by $[\oint \frac{dz}{2i\pi}(cT - bc\partial c), W_n^{(1),(2)}] = 0$.

SU(n) multiplets and their structure constants

In this section, we will demonstrate the straightforward construction of the $SU(N)$ multiplets of the ghost-dependent discrete states and compute their structure constants in the case of $N = 3$. The discrete states generated by the lowering operators of the current algebra (14),(15) realise various (generically, reducible) representations of the SU(3). We start with the decomposition of the current algebra (13),(14),(15)

$$SU(3) = N_+ \oplus N_0 \oplus N_- \tag{24}$$

with the operators L and H being in the Cartan subalgebra N_0, the subalgebra N_+ consisting of 3 operators $G_\pm$ and G_3 with the unit positive momentum and with 3 lowering operators $F_\pm$ and F_3 with the unit negative momentum being in N_-. This corresponds to the Gauss decomposition of $SL(3, C)$ which compact real form is isomorphic to SU(3). Then the full set of the $SU(3)$ multiplet states can be obtained simply by the various combinations of the N_--operators acting on the set of the highest weight vector states. So our goal now is to specify the highest weight vectors. In fact, it's easy to check that, just as in the SU(2) case, the highest weight vectors are simply the dressed tachyonic operators

$$\oint \frac{dz}{2i\pi} V_l(z) = \oint \frac{dz}{2i\pi} e^{ilX+(l-1)\varphi}(l\psi(z) - i(l-1)\lambda) \tag{25}$$

where l is integer valued, however there is one important subtlety. It is clear, for example, that the tachyon with $l = 1$ cannot be the highest weight vector since $[H, \oint \frac{dz}{2i\pi} V_1] \equiv [H, T_{0,1}] = \frac{1}{\sqrt{2}} T_{-3,1}$, therefore $\oint \frac{dz}{2i\pi} V_1$ is not an eigenvector of N_0. In addition, $\oint \frac{dz}{2i\pi} V_1$ isn't annihilated ny N_+ since $[T_{-3,1}, \oint \frac{dz}{2i\pi} V_1] \sim T_{-3,2}$. Therefore we have to examine carefully how the operators of N_0 and N_+ act on generic $\oint \frac{dz}{2i\pi} V_l$. First of all, it is clear that all the V_l's with $l \geq 2$ are annihilated by N_+ since their OPE's with the integrands of $G_\pm$ and G_3 are non-singular. Also, all these tachyons are the eigenvectors of L of N_0 with the weight $\frac{l}{2}$ i.e. one half of the isospin value (the actual isospin value must be taken equal to l since our conventions involve the factor of $-\frac{1}{2}$ in the stress-energy tensor and

hence our normalization of the $< XX >$-propagator is one half of the one used in [2];
thus the L-operator corresponds to one half of the appropriate Gell-Mann matrix). So we
only need to check how these operators are acted on by the hypercharge generator H of
N_0. Simple calculation gives

$$[H, \oint \frac{dz}{2i\pi} V_l] = [\frac{i}{3\sqrt{2}} \oint \frac{dz}{2i\pi} e^{-3\phi}(\partial^2 X\psi - 2\partial X\partial\psi), \oint \frac{dw}{2i\pi} e^{ilX}(l\psi - i(l-1)\lambda)(w)]$$

$$= \frac{i}{3\sqrt{2}} \oint \frac{dw}{2i\pi} e^{-3\phi+ilX+(l-1)\varphi} \{3il^2\partial\psi\psi - 3l\partial^2 X + \frac{1}{2}il^2 P_{-3\phi}^{(2)} \quad (26)$$

$$+(6l\partial X - 3l(l-1)\psi\lambda)\partial\phi + 3l(l-1)\partial\psi\lambda\}$$

It is convenient to integrate by parts the terms containing the derivatives of ϕ. Performing
the partial integration we get

$$[H, \oint \frac{dz}{2i\pi} V_l] = \frac{i}{3\sqrt{2}} \oint \frac{dz}{2i\pi} e^{-3\phi+ilX+(l-1)\varphi} \{3il^2\partial\psi\psi - l(1+\frac{l^2}{2})\partial^2 X$$

$$+il^2(l-1)\partial^2\varphi + l(l-1)(2\partial\psi\lambda - \psi\partial\lambda) + (il\partial X + (l-1)\partial\varphi)(2l\partial X - l(l-1)\psi\lambda) \quad (27)$$

$$+\frac{il^2}{2}(il\partial X + (l-1)\partial\varphi)^2\}$$

The operator on the right-hand side of (27) is just the dressed tachyon at the picture
-3, up to the hypercharge related numerical factor. To compute the value of the hyper-
charge we have to picture transform this operator three times to bring it to the original
zero picture. Since we are working with the integrated vertices, the c-ghost term of the
local picture-changing operator $\sim c\partial\xi$ doesn't act on the integrand (this can also be seen
straightforwardly from the $[Q_{brst}; \xi...]$-representation of the picture-changing), while the
b-ghost term annihilates the right-hand side of (27). Therefore it is sufficient to consider
the matter part of the picture- changing operator

$$\Gamma =: \delta(\beta)G_{matter} := -\frac{i}{\sqrt{2}}e^{\phi}(\psi\partial X + \lambda\partial\varphi + \partial\lambda) \quad (28)$$

Again, the factor of $\frac{1}{\sqrt{2}}$ in the matter supercurrent G_{matter} appears because our normal-
ization of the $< XX >$-propagator differs from [2] by the factor of $\frac{1}{2}$. Applying the
picture-changing operator to the right-hand side of (28) gives

$$: \Gamma :: [H, \oint \frac{dz}{2i\pi} V_l] := (-\frac{i}{\sqrt{2}})(\frac{i}{3\sqrt{2}}) \oint \frac{dz}{2i\pi} e^{-2\phi+ilX}\psi(z) \times (2 - 2l) \quad (29)$$

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i.e. the tachyon at the picture -2 Finally, applying the picture-changing to the right-hand side of (29) two times more is elementary and we obtain

$$: \Gamma :^3 : [H, \oint \frac{dz}{2i\pi} V_l(z)] = \frac{l(l-1)}{6} \oint \frac{dz}{2i\pi} V_l(z) \qquad (30)$$

i.e. we have proven that the tachyons with the integer momenta $l \geq 2$ are the highest weight vectors of $SU(3)$. The coefficient in front of V_l gives the value of the tachyon's SU(3) hypercharge

$$s(l) = \frac{l(l-1)}{6} \qquad (31)$$

Since either l or $l-1$ is even,this guarantees that possible values of the hypercharge are always the multiples of $\frac{1}{3}$. Having determined the highest weight vectors we can now easily obtain the spectrum of the physical states - the multiplets of SU(3). The vertex operators are simply given by

$$\oint \frac{dz}{2i\pi} V_{l;p_1 p_2 p_3} = F_+^{p_1} F_-^{p_2} F_3^{p_3} \oint \frac{dz}{2i\pi} V_l(z) \qquad (32)$$

with all possible integer values of p_1, p_2 and p_3 such that $p_1 + p_2 + 2p_3 \leq 2l$ This construction is in fact isomorphic to the Gelfand-Zetlin basis of the irreps of $SU(3)$, or to the tensor representations of $SU(3)$ on the T^P-spaces generated by the polynomials of degree $P = p_1 + p_2 + p_3$ in three variables x_1, x_2, x_3 spanned by $x_1^{p_1} x_2^{p_2} x_3^{p_3}$, with $x_{1,2,3}$ being the covariant vectors under $SU(3)$ [11]. In particular, the values of P can be used to label the irreducible representations. In our case, the values of p_1, p_2, p_3 and P can be easily related to the isospin projection m, the hypercharge s and the ghost cohomology numbers N of the vertex operators (32). Applying L and H to the operators (32) and using the commutation relations (14) we get

$$s = p_1 - p_2 + \frac{l(l-1)}{6}$$
$$m = l - p_1 - p_2 - 2p_3 \qquad (33)$$
$$N_{max} = -3P = -3(p_1 + p_2 + p_3)$$

where $N^{max} = -3P$ is the biggest (in terms of the absolute value) ghost cohomology number of the operators appearing in the expression for $V_{l;p_1 p_2 p_3}$.

Conversely,

$$p_1 = -\frac{1}{2}(m - l + s) - \frac{1}{3} N^{max} - \frac{1}{12} l(l-1)$$
$$p_2 = \frac{1}{2}(m - l - s) - \frac{1}{3} N^{max} + \frac{1}{12} l(l-1) \qquad (34)$$
$$p_3 = l - m + \frac{1}{3} N^{max}$$

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The SU(3) symmetry then makes it straightforward to determine the structure constants of the operators (32), given by the three-point correlators

$$A(l_{1,2,3}; p_{1,2,3}; q_{1,2,3}; r_{1,2,3}) =< cV_{l_1;p_1p_2p_3} cV_{l_2;q_1q_2q_3} cV_{l_3;r_1r_2r_3} > \tag{35}$$

The symmetry determines these correlators up to the overall isospin function $f(l_1, l_2)$ which we will fix later. As for the m and s-dependences of the correlator, they are completely fixed by the $SU(3)$ symmetry and, accordingly, by the $SU(3)$ Clebsch-Gordan coefficients. It is now straightforward to deduce the structure constants by using the relations (33),(34) and the SU(3) Clebsch-Gordan decomposition:

$$|P; lms >= \sum_{\{Q,R;|Q-R|\leq P\}} \sum_{\{l_1,l_2;|l_1-l_2|\leq l\}} \sum_{\{\mu,\sigma\}}$$

$$D^{Q;R|P}_{l_1\mu\sigma;l_2,m-\mu,s-\sigma|lms} \times |Q; l_1\mu\sigma > |R; l_2(m-\mu)(s-\sigma) >$$

$$P = p_1 + p_2 + p_3 \tag{36}$$

$$Q = q_1 + q_2 + q_3$$

$$R = r_1 + r_2 + r_3$$

where $|... >$ are the eigenfunctions of the isospin, its projection and of the hypercharge, $D^{Q;R|P}_{l_1,\mu,\sigma;l_2,m-\mu,s-\sigma|lms}$ are the $SU(3)$ Clebsch-Gordan coefficients. Furthermore, it is convenient to write the SU(3) Clebsch-Gordan coefficients in the form:

$$D^{Q;R|P}_{l_1,\mu,\sigma;l_2,m-\mu,s-\sigma} = \alpha^{l_3,s;R,P,Q}_{\sigma;l_1,l_2} \times C^{l_1,l_2,l_3}_{\mu,m-\mu,m} \tag{37}$$

where $C^{l_1l_2l_3}_{\mu,m-\mu,m}$ are the usual SU(2) Clebsch-Gordan coefficients and $\alpha^{l_3sRPQ}_{\sigma l_1l_2}$ are the $U(1)$ isoscalar factors computed in [12]. The three-point amplitude (35) is then given by

$$A(l_{1,2,3}; p_{1,2,3}; q_{1,2,3}; r_{1,2,3}) = f(l_1, l_2) D^{P,Q|R}_{l_1,m_1,s_1;l_2m_2s_2|l_3,m_1+m_1,s_1+s_2} = f(l_1, l_2)$$

$$\times \alpha^{l_3,(p_1-p_2+q_1-q_2+\frac{1}{6}(l_1(l_1-1)+l_1(l_2-1)));R,P,Q}_{p_1-p_2+\frac{l_1(l_1-1)}{6};\frac{l_1}{2},l_2} C^{l_1,l_2,l_3}_{l_1-p_1-p_2-2p_3,l_2-q_1-q_2-2q_3,l_3-r_1-r_2-2r_3} \tag{38}$$

where

$$P = p_1 + p_2 + p_3 = -\frac{1}{3}N^{max}_1$$

$$Q = q_1 + q_2 + q_3 = -\frac{1}{3}N^{max}_2 \tag{39}$$

$$R = r_1 + r_2 + r_3 = -\frac{1}{3}N^{max}_3$$

$$l_1 + l_2 - p_1 - p_2 - 2p_3 - q_1 - q_2 - 2q_3 = l_3 - r_1 - r_2 - 2r_3$$

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The final step is to determine the function $f(l_1, l_2)$. To find it we will follow the procedure quite similar to [2] and consider the special case of p, q and r when the computation of the OPE can be done explicitly; then we will compare the result with the general formula (38) in order to fix $f(l_1, l_2)$. Namely, consider the OPE

$$V_{l_1;010}(z)V_{l_2;000}(w) =: F_- e^{il_1 X+(l_1-1)\varphi} : (z) : e^{il_2 X+(l_2-1)\varphi} : (w) \qquad (40)$$

To evaluate the structure constants of this OPE we note that the all the operators appearing on the right-hand side carry the momentum $p_x = l_1 + l_2 - 1$ in the X-direction and $p_L = l_2 + l_1 - 2$ in the Liouville direction. But the only primary field of dimension 1 with such a property is (up to the picture-changing) the highest weight vector (25) with $l = l_1 + l_2 - 1$. Therefore we have

$$V_{l_1;010}(z)V_{l_2;000}(0) \sim \frac{A}{z} V_{l_1+l_2-1;000}(0) \qquad (41)$$

and we only need to determine the structure constant A in order to fix $f(l_1, l_2)$. On the left hand side, it is convenient to take the highest weight vectors at the ghost picture -1:

$$V_{l;000} \equiv V_l = -\sqrt{2}e^{-\phi+ilX+(l-1)\varphi}; l = l_1, l_2$$

where the normalization factor of $-\sqrt{2}$ ensures that applying the picture-changing operator Γ to $V_{l;000}$ at picture -1 gives $V_{l;000}$ at picture zero normalized according to (25). As $F_- = \frac{1}{2}T_{0,-1} - \frac{1}{2\sqrt{2}}T_{-3,-1}$, we start with the evaluation of the OPE of the $: T_{0,-1}e^{-\phi+il_1 X+(l_1-1)\varphi} :$ part of $V_{l_1;100}$ with $V_{l_2;000}$. We have:

$$: T_{0,-1}V_{l_1;000} : (z)V_{l_2;000}(0)$$

$$= 2 \oint \frac{du}{2i\pi} : e^{-iX}\psi : (u+z) : e^{-\phi+il_1 X+(l_1-1)\varphi} : (z)e^{-\phi+il_2 X+(l_2-1)\varphi}(0) \qquad (42)$$

$$= e^{-2\phi+i(l_1+l_2-1)X+(l_1+l_2-2)\varphi}\psi(0) \oint \frac{du}{2i\pi}u^{-l_1}(u+z)^{-l_2}z^{l_1+l_2-2}$$

Introducing the integration variable $w = \frac{u}{z}$ have:

$$2 \oint \frac{du}{2i\pi}u^{-l_1}(u+z)^{-l_2}z^{l_1+l_2-2} = \frac{2}{z} \oint \frac{dw}{2i\pi}w^{-l_1}(1+w)^{-l_2} = \frac{2}{z} \times \frac{(l_1+l_2-2)!}{(l_1-1)!(l_2-1)!} \qquad (43)$$

Transforming the right-hand side to the zero picture by the operator (28) and noting that

$$: \Gamma^2 : \oint \frac{dz}{2i\pi}e^{-2\phi+i(l_1+l_2-1)X+(l_1+l_2-2)\varphi}\psi(z) = -\frac{l_1+l_2-1}{2} \oint \frac{dz}{2i\pi}V_{l_1+l_2-1;000}(z)$$

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we get:

$$: T_{0,-1}V_{l_1;000} : (z) : V_{l_2;000} : (0) \sim -\frac{1}{z}\frac{(l_1 + l_2 - 1)!}{(l_1 - 1)!(l_2 - 1)!}V_{l_1+l_2-1;000}(0) \qquad (44)$$

The next step is to compute the OPE between $T_{-3;-1}V_{l_1;000}$ and $V_{l_2;000}$.

To simplify the calculation, we shall take the generator $T_{-3;-1}$ at the picture $+1$ by using the equivalence relations (12) and the operators $V_{l_1;000}$ and $V_{l_2;000}$ both at the picture -1 : $V_{l_{1,2};000} = -\sqrt{2}e^{-\phi+il_{1,2}X+(l_{1,2}-1)\varphi}$. Note that, as the expressions the operators $V_{l_{1,2};000}$ at the picture -1 do not contain any b and c ghost fields, we can as well disregard the $b - c$ ghost part of $T_{+1,1}$ (since the form of the OPE (41) is fixed, the interaction of this part with the highest weight vectors can only contribute BRST-trivial terms)

We have:

$$: T_{+1;1}V_{l_1;000} : (z)V_{l_2;000}(0)$$

$$= 2\oint \frac{du}{2i\pi}e^{\phi-iX}\{\partial\psi\psi + \frac{1}{2}(\partial X)^2 - \frac{i}{2}\partial^2 X\}(u)e^{-\phi+il_1X+(l_1-1)\varphi}(z)e^{-\phi+il_2X+(l_2-1)\varphi}(0)$$

$$= 2e^{-\phi+i(l_1+l_2-1)X+(l_1+l_2-2)\varphi}(0)\oint \frac{du}{2i\pi}u^{-l_1+1}(u+z)^{-l_2+1}z^{l_1+l_2-2}$$

$$\times\{\frac{l_1 - l_1^2}{2u^2} + \frac{l_2 - l_2^2}{2(u+z)^2} - \frac{l_1l_2}{u(z+u)}\}$$

$$= -\sqrt{2}e^{i(l_1+l_2-1)X+(l_1+l_2-2)\varphi}((l_1 + l_2 - 1)\psi - i(l_1 + l_2 - 2)\lambda)(0)$$

$$\times \oint \frac{du}{2i\pi}u^{-l_1+1}(u+z)^{-l_2+1}z^{l_1+l_2-2}\{\frac{l_1 - l_1^2}{2u^2} + \frac{l_2 - l_2^2}{2(u+z)^2} - \frac{l_1l_2}{u(z+u)}\}$$

$$(45)$$

where the terms inside the figure brackets in the integral are due to the contractions of the derivatives of X in the expression for $T_{1,-1}$ with the exponents in $V_{l_{1,2};000}$ (note that the term with $\partial\psi\psi$ of $T_{1,-1}$ cannot contribute anything but the BRST-trivial part because of the fixed form of the OPE (41)) Performing the contour integration precisely is in (42)-(44) we obtain

$$: T_{1,1}V_{l_1;000} : (z)V_{l_2;000}(0) = -\frac{\sqrt{2}}{z}V_{l_1+l_2-1;000}$$

$$\times\{\frac{(l_1 - l_1^2)}{2}\frac{(l_1 + l_2 - 1)!}{l_1!(l_2 - 2)!} + \frac{(l_2 - l_2^2)}{2}\frac{(l_1 + l_2 - 1)!}{l_2!(l_1 - 2)!} - l_1l_2\frac{(l_1 + l_2 - 2)!}{(l_1 - 1)!(l_2 - 1)!}\} \qquad (46)$$

$$= -\frac{\sqrt{2}}{z}V_{l_1+l_2-1;000} \times \frac{(l_1 + l_2 - 2)!}{(l_1 - 1)!(l_2 - 1)!}(l_1 + l_2 - 1 - l_1l_2)$$

Finally, collecting together (44) and (46) we get

$$F_-V_{l_1;000}(z)V_{l_2;000}(0) = -\frac{l_1l_2}{2z}\frac{(l_1 + l_2 - 2)!}{(l_1 - 1)!(l_2 - 1)!}V_{l_1+l_2-1;000} \qquad (47)$$

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Comparing the operator product (47) with the general expression (38) for $q_{1,2,3} = r_{1,2,3} = 0$, $p_1 = p_3 = 0$ and $p_2 = 1$ we can now easily fix the isospin function $f(l_1, l_2)$ to be equal to

$$
\begin{aligned}
f(l_1, l_2) &= -\frac{l_1 l_2}{2} \frac{(l_1 + l_2 - 2)!}{(l_1 - 1)!(l_2 - 1)!} (\alpha_{\frac{l_1(l_1-1)}{6}-1;l_1,l_2}^{l_1+l_2-1,\frac{l_1(l_1-1)+l_2(l_2-1)}{6};0,1,0} \times C_{l_1-1,l_2,l_1+l_2-1}^{l_1,l_2,l_1+l_2-1})^{-1} \\
&= \frac{1}{2} l_1 l_2 \sqrt{\frac{l_1 + l_2}{l_2}} \frac{(l_1 + l_2 - 2)!}{(l_1 - 1)!(l_2 - 1)!} (\alpha_{\frac{l_1(l_1-1)}{6}-1;l_1,l_2}^{l_1+l_2-1,\frac{l_1(l_1-1)+l_2(l_2-1)}{6};0,1,0})^{-1}
\end{aligned}
\tag{48}
$$

This concludes the computation of the structure constants for the ghost-dependent discrete states of SU(3) multiplet.

Conclusion and discussion

In this paper we have shown that the interaction of the 2-dimensional supergravity with the $\beta - \gamma$ system of the superconformal ghost crucially extends the spectrum of the physical states and enhances the underlying symmetry of the theory. We have shown that the discrete states generated by the currents of the ghost number cohomologies such that $-N \leq n \leq N - 2; N \geq 3$,combined with the standard SU(2) discrete states, form the $SU(N)$ multiplets. We have demonstrated this explicitly for $N = 3$ and 4 and conjectured for higher values of N.

In the $N = 3$ case structure constants of the ghost-dependent discrete states of the SU(3) multiplet are determined by the appropriate Klebsch-Gordan coefficients. Since $SU(3)$ is isomorphic to $SL(3, R)$, the operator algebra (38), (48) is related to the algebra of the volume preserving diffeomorphisms in 3 dimensions after the appropriate rescaling of the vertices, just as the OPEs of the standard $SU(2)$ discrete states lead the algebra of the area preserving diffeomorphisms on the plane after the redefinition of the fields [2]. Physically, the volume preserving diffeomorphisms $sDiff(3)$ in three dimensions describe the dynamics of ideal incompressible fluid in $d = 3$. In this context, the appearance of the CG coefficients in the structure constants is quite natural, as the vertex operators (32) can be interpreted as Clebsch variables for the space of vorticities [13], [14]. The beta-functions of the $SU(3)$ multiplet states would then generate the RG flows similar to the dynamics of the co-adjoint orbits considered in [14]. From this point of view, the RG equations are the Lie-Poisson equations associated to $sDiff(3)$. Unfortunately, because of the complexity of the expressions for the SU(3) CG coefficients, as well as due to difficulties in the classification of the volume preserving diffeomorphisms in higher dimensions, the precise form of the necessary field rescaling appears to be far more complicated then in the $SU(2)$ case. We hope to elaborate on it in details in the future work.

Generalizing these arguments to the ghost cohomologies of higher ghost numbers, it is seems natural to conjecture that the structure constants of the $SU(N)$ multiplets are related to the Clebsch-Gordan coefficients of SU(N), up to the functions of the Casimir eigenvalues which should be determined similarly to the isospin function $f(l_1, l_2)$ of (48), by considering the special cases of the operators when straightforward computation of the OPEs is accessible. The operator algebra should be related then to the volume preserving diffeomorphisms in N dimensions and the vertex operators of the $SU(N)$ multiplet could be interpreted in terms of the Clebsch variables of the vortices in N-dimensional incompressible liquid. The case of the special interest is $N = 4$; the ghost-dependent discrete states are then the $SU(4)$ multiplets, i.e. are in the representation of the conformal group in $d = 4$. The currents generating the $SU(4)$ algebra could then be related to $d = 4$ conformal generators. It would be interesting to relate them to the twistor variables of $d = 4$ twistor superstrings [15], [16]. Particularly, the physical vertices of $d = 2$ closed superstrings with the holomorphic and antiholomorphic parts being in the ghost cohomologies of negative numbers $-n$ with $|n| \leq 4$, combined with the standard $SU(2) \oplus SU(2)$ primaries, are the multiplets of $SU(4) \oplus SU(4)$, isomorphic to the bosonic part of $PSU(2,2|4)$, the full symmetry group of open twistor superstrings and of the $d = 4$ super Yang-Mills theory. All this implies that the $c = 1$ theory coupled to the $\beta - \gamma$ ghosts can be used to describe field theories in higher dimensions. The question is where the extra dimensions come from. This question will be studied in details in our next paper, while in this work we shall give only the rough qualitative explanation. In order to answer this question one has to carefully analyze the β-functions of the $SU(N)$ multiplets. The important subtlety here is that the structure constants computed in (38), (48), do not fully determine the β-functions, as they involve only the states from the ghost cohomologies of negative numbers. However, because of the equivalence relations (12) there are also the positive ghost number representations for these states. That is, the relations (12) imply the isomorphism between the ghost cohomologies of negative and positive numbers:

$$G_{-N} \sim G_{N-2} \qquad (49)$$

One consequence of this isomorphism is that the OPEs of the ghost-dependent discrete states are, generally speaking, picture-dependent. The illustrate the picture dependence of the OPEs, consider the following simple example. Consider the OPE of two operators $V_1 = \oint \frac{dz}{2i\pi} W_1(z)$ and $V_2 = \oint \frac{dz}{2i\pi} W_2(z)$ in the ghost number cohomologies $-N_1 \leq -3$ and

$-N_2 \leq -3$. For simplicity,let us take them in their maximal existing pictures that is, $-N_1$ and $-N_2$ and consider the special case of $N_2 - N_1 \geq 3$). By using the equivalence relations (12) similar to the reflection identities of the Liouville theory [9], [10], let us first take V_1 at the *positive* picture $N_1 - 2$ and leave V_2 at the $-N_2$-picture (recall that V_1 exists in all the pictures below $-N_1$ and all above $N_1 - 2$; similarly for V_2). The necessary equivalence transformation is given by

$$V_1 \rightarrow Z(: \Gamma^{2N_1-2}cW_1 :) \tag{50}$$

Then the OPE of V_1 and V_2 consists of the operators of ghost number cohomologies up to $G_{N_1-N_2-2} \sim G_{N_2-N_1}$ (given the equivalence relation (12)) Conversely, let us take V_2 at the picture $N_2 - 2$ by the equivalence transformation

$$V_2 \rightarrow Z(: \Gamma^{2N_2-2}cW_2 :) \tag{51}$$

and leave V_1 at $-N_1$ The OPE of V_1 and V_2 would then consist of operators from the ghost number cohomologies up to $G_{N_2-2-N_1} \sim G_{N_1-N_2}$ Thus, unlike the first OPE, the latter operator product skips the operators of the ghost number cohomologies $G_{N_2-N_1-1} \sim G_{N_1-N_2-1}$ and $G_{N_2-N_1} \sim G_{N_1-N_2-2}$. These operators are simply screened off by the extra powers of Γ in the equivalence transformation for V_2. In the functional integral (which sums over all the ghost pictures) such a picture assymmetry would generally lead to the Non-Markovian stochastic terms,or the explicit dependence of the β-function on the operator-valued worldsheet variables, as it has been explained in [17]. Typically, such β-function equations have the form of either the Langevin stochastic equations, or of equations for incompressible fluid with the random force [17]. The role of the noise is played by the worldsheet integrals of ghost-dependent vertex operators cut of at a scale Λ. In analogy with the stochastic quantization approach, one can interpret the effective stochastic time Λ as the extra dimension of the theory. Roughly speaking, the equations of the RG flows induced by the ghost-dependent states of $SU(N)$ multiplet ($N \geq 3$), describe the non-Markovian process with $N-2$ stochastic times, effectively bringing the $N-2$ extra dimensions to the theory. In other words, each new ghost cohomology of the underlying current algebra of the T-operators brings an extra stochastic time in the Non-Markovian RG flow and, subsequently, an extra dimension. This qualitatively explains the appearance of the extra dimensions but of course at this point the explanation is still heuristic and the entire question is yet to be explored in details.

We conclude this paper by noting that the operator algebra (38),(48) particularly entails some remarkable relations concerning the decomposition rules for the ghost cohomologies of negative numbers. Consider again the product of V_1 and V_2 discussed above. All of the operators on the right hand side of this OPE have the same ghost number $-N_1 - N_2$ but, generally speaking, come from very different ghost cohomologies. The relations (38),(48) thus define the precise decomposition rules for the product of two cohomologies. As we have shown, these decomposition rules are regulated by $SU(N)$ Clebsch-Gordan coefficients, the fact pointing at the underlying relations between ghost numbers and spaces of vorticities. In this context, the ghost cohomologies may be useful for the classification of vortices in an incompressible fluid.

Acknowledgements

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